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On Stability of Some General Random Dynamical System

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We consider a new random dynamical system which generalizes Markov processes corresponding to iterated function systems and Poisson driven stochastic differential equations. It can be used as a description of many physical and biological phenomena. Under the suitable assumption will be proved its stability.

KEY WORDS: Dynamical systems; Markov operators; stability.

1. INTRODUCTION

In this paper we propose a new model generalizing Poisson stochastic differential equations and iterated function systems. A Poisson process is one of the fundamental descriptions for physical and biological phenomena—these phenomena are generally described by stochastic differential equations with Poisson drift rather than the Wiener drift (see refs. 4, 7, 14, 26). In fact, Wiener processes may be obtained when we pass to the limit with intensivity of the Poisson process. However, it seems that we obtain more realistic models with Poisson drift. A large class of applications of such models, both in physics and biology, is worth mentioning here: the short noise, the photoconductive detectors, the growth of the size of structural population, the motion of relativistic particles, both fermions and bosons, and many others (see refs. 7, 14, 16, 17, 26). In 1984 Gaveau

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et al.[8] derived the Dirac equation using the formulation of Poisson process.

On the other hand, it should be noted that most Markov chains, appear among other things, in statistical physics, and may be represented as iterated function systems (see ref. 15). They are also intensively studied as a mathematical model of learning and random walks and have turned out to be a very useful tool in the theory of cell cycles (see refs. 18, 19, 29 and references therein). Recently, iterated function systems have been used in studying invariant measures for the Ważewska partial differential equation which describes the process of reproduction of the red blood cells (see ref. 21). Similar nonlinear first-order partial differential equations frequently appear in hydrodynamics (see ref. 28).

Today iterated function systems are considered mainly because of their close connection to fractals and semifractals. Indeed, a fractal set (analogously semifractal) may be obtained as a support of an invariant measure for such systems (see refs. 19, 20).

We formulate criterion for stability. There is excellent literature devoted to such problems (see ref. 25). Different classes of Markov processes have been studied e.g. random dynamical systems based on skew product flows and piecewise-deterministic Markov processes introduced by Davis (see refs. 1, 3). Our model generalizes the latter. Besides physics and biology there is an enormous variety of their applications in engineering systems, operation research, management science, economics and applied probability (for more details see ref. 3 and references therein).

Usually the proof of stability is based on the theory of Meyn and Tweedie presented in ref. 25 which, to the best of our knowledge, is not well adapted to general Banach spaces. In fact, it is extremely difficult to ensure that the process under consideration satisfies some ergodic properties on a compact set. However assumption of compactness is restrictive if we want to apply our model in physics and biology. Indeed, the phase space is usually one of the spaces of functions and the above assumption is therefore not satisfied. In our paper, we apply the theory of concentrating Markov operators developed in ref. 29.

Let $(X, \|\cdot\|)$ be a separable Banach space. Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and let $(\tau_n)_{n\geqslant 0}$ be a sequence of random variables $\tau_n\colon \Omega \to \mathbb{R}_+$ with $\tau_0 = 0$ and such that the increments $\Delta\tau_n = \tau_n - \tau_{n-1}$, $n \in \mathbb{N}$, are independent and have the same density $g(t) = \lambda e^{-\lambda t}$.

We have given a finite sequence of semidynamical systems $\Pi_i\colon \mathbb{R}_+ \times X \to X$, $i \in I = \{1, \dots, N\}$, a probability vector $p_i\colon X \to [0, 1]$, $i \in I$ and a matrix of probabilities $[p_{ij}]_{i,j\in I}$, $p_{ij}\colon X \to [0, 1]$, $i, j \in I$.

Let S be a compact metric space. By $\mathcal{F}$ we define the Borel σ-algebra on S. Let $(\zeta_n)_{n\geqslant 0}$ be a sequence of random elements $\zeta_n:\Omega\to S$, $n\in\mathbb{N}$ with the same distribution κ, i.e. $\kappa(A)=\mathbb{P}\big(\zeta_n^{-1}(A)\big)$ for $A\in\mathcal{F}$ and $n\in\mathbb{N}$. Obviously $\kappa(S)=1$. We assume that $(\zeta_n)_{n\geqslant 0}$ is independent on $(\tau_n)_{n\geqslant 0}$. Finally let $q:S\times X\to X$ be a continuous function. We write $q_s=q(s,\cdot)$ for $s\in S$.

Now we define X-valued stochastic process $(\xi_n)_{n\geqslant 0}$ in the following way. We choose an initial point $x\in X$ and we randomly select an integer $i\in I$ in such a way that probability of choosing i is equal to $p_i(x)$. Having x and i we define

$$\xi_1=q_{\zeta_1}\big(\Pi_i(\tau_1,x)\big).$$

Now we choose $i_1\in I$ in such a way that the probability of chosing i_1 is equal to $p_{ii_1}(\xi_1)$ and is independent upon the variable ζ_1. Then we define

$$\xi_2=q_{\zeta_2}\big(\Pi_{i_1}(\tau_2-\tau_1,\xi_1)\big).$$

Finally, given ξ_n, $n\geqslant 2$, we choose i_n in such a way that the probability of choosing i_n is equal to $p_{i_{n-1}i_n}(\xi_n)$ and is independent upon $\zeta_1,\dots,\zeta_n$, $\xi_1,\dots,\xi_n$. Then we define

$$\xi_{n+1}=q_{\zeta_{n+1}}\big(\Pi_{i_n}(\tau_{n+1}-\tau_n,\xi_n)\big).$$

We are interested in the evolution of distributions corresponding to this random dynamical system. Namely, let μ be the distribution of the initial random vector x. For $n\in\mathbb{N}$ we denote by μ_n the distribution of ξ_n, i.e.

$$\mu_n(A)=\mathbb{P}\big(\xi_n\in A\big)=\int_X \mathbb{P}\big(\xi_n(x)\in A\big)\mu(dx),$$

where $\xi_n(x)$ means the process started from the initial point x and A is an arbitrary Borel set in X. We will prove that there exists a distribution μ_* on X such that $\mu_n\to\mu_*$ (weakly) as $n\to\infty$ for arbitrary initial distribution μ on X.

The examples below show that our model generalizes some very important and widely studied objects, namely dynamical systems generated by iterated function systems and Poisson driven stochastic differential equations.

Example 1. Iterated funcion systems.

Let X be a separable Banach space, $w_i : X \to X$, $i \in I$, continuous functions and let $(p_1, \dots, p_N)$ be a probability vector, i.e. $p_i > 0$, $\sum_{i\in I} p_i = 1$.

Assume that $\Pi_i(t,x) = x$ for $i \in I$, $t \in \mathbb{R}_+$ and $x \in X$. Moreover, assume that $S = I$, $\mathcal{F} = 2^I$ and $\kappa(\{i\}) = p_i$. Finally, assume that $q(i,x) = w_i(x)$ for $i \in I$ and $x \in X$.

Let μ be the distribution of the initial random vector x. Simple calculation shows that the distribution μ_n of the random vector ξ_n is given by

$$\mu_n(A) = \sum_{i_1,\dots,i_n \in I} p_{i_1} \cdot \dots \cdot p_{i_n} \mu\left(\left(w_{i_n} \circ \dots \circ w_{i_1}\right)^{-1}(A)\right).$$

But this means that $\mu_n = P^n \mu$, where P is the well known (see refs. 18, 19, 29) transition operator corresponding to iterated function system $\{(w_i, p_i); i \in I\}$. It is of the form

$$P\mu = \sum_{i=1}^{N} p_i \mu \circ w_i^{-1}.$$

Example 2. Poisson driven stochastic differential equations.

Consider a stochastic differential equation of the form

$$d\xi = a(\xi)dt + b(\xi)dp \quad \text{for} \quad t > 0$$

with the initial condition

$$\xi(0) = \xi_0,$$

where $a, b : X \to X$ are Lipschitzian functions, X is a separable Banach space, $\left(p(t)\right)_{t\geqslant 0}$ is a Poisson process and the initial condition ξ_0 is a random variable on Ω with values in X, independent on $\left(p(t)\right)_{t\geqslant 0}$.

Let $S = I = \{1\}$ and let $\Pi_1(t,x) = \Pi(t,x)$ be the unique solution of the Cauchy problem

$$\frac{du}{dt} = a(u(t)), \quad u(0) = x.$$

Moreover, let $q_1(x) = q(x) = x + b(x)$. It is easy to check that $\mu_n = P^n\mu$, $n \in \mathbb{N}$, where P is the transition operator corresponding to the above stochastic equation and given by

$$P\mu(A) = \int_X \int_{\mathbb{R}_+} \lambda e^{-\lambda t} 1_A(q(\pi(t,x)))dt\mu(dx).$$

(Here 1_A stands for the characteristic function of A.)

The paper is divided into five sections. Section 2 contains further notation and some known facts concerning asymptotic stability of Markov operators crucial for our considerations. In Section 3 we state all necessary hypotheses and reformulate our problem in a more convenient form. Section 4 contains some technical lemmas. The main results are contained in the last section. For related results see refs. 1–3, 10–13, 18, 22–24, 27, 29–31. The basic facts on Markov processes and stochastic differential equations can be found in refs. 2, 5, 7, 19.

2. NOTATION AND SOME USEFUL FACTS

Let $(\mathfrak{X}, \varrho)$ be a complete separable metric space. By $B(x,r)$ we denote the open ball with center at x and radius r. For a subset A of $\mathfrak{X}$, $\mathrm{cl}\,A$, $\mathrm{diam}\,A$, and 1_A stands for the closure of A, diameter of A and the characterisic function of A, respectively.

By $\mathcal{B}(\mathfrak{X})$ we denote the σ-algebra of Borel subsets of $\mathfrak{X}$ and by $\mathcal{M} = \mathcal{M}(\mathfrak{X})$ the family of all finite Borel measures on $\mathfrak{X}$. By $\mathcal{M}_1 = \mathcal{M}_1(\mathfrak{X})$ we denote the space of all $\mu \in \mathcal{M}$ such that $\mu(\mathfrak{X}) = 1$ and by $\mathcal{M}_s$ the space of all finite signed Borel measures on $\mathfrak{X}$. The elements of $\mathcal{M}_1$ are called *distributions*.

As usual, by $B(\mathfrak{X})$ we denote the space of all bounded Borel measurable functions $f: \mathfrak{X} \to \mathbb{R}$ and by $C(\mathfrak{X})$ the subspace of all continuous functions. Both spaces are considered with the supremum norm $\|\cdot\|_0$.

For $f \in B(\mathfrak{X})$ and $\mu \in \mathcal{M}_s$ we write

$$< f, \mu > = \int_{\mathfrak{X}} f(x)\mu(dx).$$

We introduce in $\mathcal{M}_s$ the *Fortet–Mourier norm* $\|\cdot\|_\varrho$ (see ref. 6) given by

$$\|\mu\|_\varrho = \sup\{|< f, \mu >| : f \in \mathcal{F}_\varrho\} \quad \text{for} \quad \mu \in \mathcal{M}_s,$$

where $\mathcal{F}_\varrho$ is the set of all $f \in C(\mathfrak{X})$ such that $|f(x)| \leqslant 1$ and $|f(x) - f(y)| \leqslant \varrho(x,y)$ for $x, y \in \mathfrak{X}$.

We say that a sequence (μ_n), $\mu_n \in \mathcal{M}$, converges weakly to a measure $\mu \in \mathcal{M}$ if

$$\lim_{n\to\infty} < f, \mu_n > = < f, \mu > \quad \text{for every} \quad f \in C(\mathfrak{X}).$$

It is well known (see ref. 5) that the convergence in the Fortet–Mourier norm $\|\cdot\|_\varrho$ is equivalent to the weak convergence.

An operator $P: \mathcal{M} \to \mathcal{M}$ is called a *Markov operator* if

$$P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2 \quad \text{for} \quad \lambda_1,\ \lambda_2 \in \mathbb{R}_+ \quad \text{and} \quad \mu_1,\ \mu_2 \in \mathcal{M}$$

and

$$P\mu(\mathfrak{X}) = \mu(\mathfrak{X}) \quad \text{for} \quad \mu \in \mathcal{M}.$$

A linear operator $U: B(\mathfrak{X}) \to B(\mathfrak{X})$ is called *dual* to P if

$$< Uf, \mu > = < f, P\mu > \quad \text{for} \quad f \in B(\mathfrak{X}) \quad \text{and} \quad \mu \in \mathcal{M}.$$

A Markov operator P is called a *Feller* operator if it has a dual operator U such that

$$Uf \in C(\mathfrak{X}) \quad \text{for} \quad f \in C(\mathfrak{X}).$$

An operator $P: \mathcal{M} \to \mathcal{M}$ is called *essentially nonexpansive* if there exists a metric $\hat{\varrho}$ equivalent to ϱ such that P is nonexpansive with respect to the norm $\|\cdot\|_{\hat{\varrho}}$, i.e.

$$\|P\mu_1 - P\mu_2\|_{\hat{\varrho}} \leqslant \|\mu_1 - \mu_2\|_{\hat{\varrho}} \quad \text{for} \quad \mu_1, \mu_2 \in \mathcal{M}_1.$$

It can be proved that every essentially nonexpansive Markov operator is a Feller operator (see ref. 29).

A measure μ_* is called *invariant* (or *stationary*) with respect to P if $P\mu_* = \mu_*$. A Markov operator P is called *asymptotically stable* if there exists a stationary measure $\mu_* \in \mathcal{M}_1$ such that

$$\lim_{n\to\infty} P^n \mu = \mu_* \quad \text{for every} \quad \mu \in \mathcal{M}_1.$$

Obviously a measure μ_* satisfying the above condition is unique.

A sequence of distributionss (μ_n) is called *tight* if for every $\varepsilon > 0$ there exists a compact set $K \subset \mathfrak{X}$ such that $\mu_n(K) \geqslant 1 - \varepsilon$ for every $n \in \mathbb{N}$.

It is well known (see refs. 2, 5) that every tight sequence of distributions contains a weakly convergent subsequence.

We say that a Markov operator $P: \mathcal{M} \to \mathcal{M}$ is *tight* if for every $\mu \in \mathcal{M}_1$ the sequence of iterates $(P^n \mu)$ is tight.

We denote by $\mathcal{C}_\varepsilon(\mathfrak{X})$, $\varepsilon > 0$, ($\mathcal{C}_\varepsilon$ for abbreviation), the family of all closed sets C for which there exists a finite set $\{x_1, x_2, \ldots, x_n\} \subset \mathfrak{X}$ (ε-net) such that $C \subset \bigcup_{i=1}^n B(x_i, \varepsilon)$.

An operator P is called *semi-concentrating* if for every $\varepsilon > 0$ there exist $C \in \mathcal{C}_\varepsilon(\mathfrak{X})$ and $\theta > 0$ such that

$$\liminf_{n \to \infty} P^n \mu(C) > \theta \quad \text{for} \quad \mu \in \mathcal{M}_1. \tag{2.1}$$

Proposition 2.1 (ref. 23). Let P be a nonexpansive Markov operator. Assume that for every $\epsilon > 0$ there exists a number $\theta > 0$ having the following property: for every pair of measures $\mu_1, \mu_2 \in \mathcal{M}_1$ there exist a Borel subset A of $\mathfrak{X}$ with $\operatorname{diam} A \leqslant \epsilon$ and a number $n_0 \in \mathbb{N}$ such that

$$P^{n_0} \mu_k(A) > \theta \quad \text{for} \quad k = 1, 2.$$

Then

$$\lim_{n \to \infty} \| P^n \mu_1 - P^n \mu_2 \|_\varrho = 0 \quad \text{for every} \quad \mu_1, \mu_2 \in \mathcal{M}_1.$$

For $\mu \in \mathcal{M}_1$ we consider the limit set:

$$\mathcal{L}(\mu) = \Big\{ \nu \in \mathcal{M}_1 : \text{there exists } (n_k) \subset (n) \text{ such that } \lim_{k \to \infty} \| P^{n_k} \mu - \nu \|_\varrho = 0 \Big\} \tag{2.2}$$

and

$$\mathcal{L}(\mathcal{M}_1) = \bigcup_{\mu \in \mathcal{M}_1} \mathcal{L}(\mu). \tag{2.3}$$

Proposition 2.2 (ref. 29). Let P be a nonexpansive and semi-concentrating Markov operator. Then

(i) P has an invariant measure;
(ii) $\mathcal{L}(\mu) \neq \emptyset$ for arbitrary $\mu \in \mathcal{M}_1$;
(iii) $\mathcal{L}(\mathcal{M}_1)$ is tight.

Finally, we introduce the class Φ of functions $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:

(i) φ is continuous and $\varphi(0) = 0$;
(ii) φ is nondecreasing and concave;
(iii) $\varphi(x) > 0$ for $x > 0$ and $\lim_{x\to\infty} \varphi(x) = \infty$.

By Φ_0 we denote the family of all functions satisfying conditions (i) and (ii). Observe that for every $\varphi \in \Phi$ the function $\rho_\varphi = \varphi \circ \rho$ is again a metric on $\mathfrak{X}$. Moreover ρ_φ is equivalent to ρ. For notational convenience we write $\mathcal{F}_\varphi$ and $\|\cdot\|_\varphi$ in the place of $\mathcal{F}_{\varrho_\varphi}$ and $\|\cdot\|_{\varrho_\varphi}$, respectively.

Proposition 2.3 (ref. 23). Assume that a function $w \in \Phi_0$ satisfies the Dini condition

$$\int_0^\epsilon \frac{w(t)}{t} dt < \infty \quad \text{for some} \quad \epsilon > 0. \tag{2.4}$$

Let $a \in [0, 1)$. Then the inequality

$$w(t) + \varphi(at) \leqslant \varphi(t) \quad \text{for} \quad t \geqslant 0 \tag{2.5}$$

admits a solution of Φ.

3. ASSUMPTIONS AND REFORMULATION OF THE PROBLEM

Assume that we have given the system (Π, q, p) on a separable Banach space defined in Section 1. Recall that $\Pi_i : \mathbb{R}_+ \times X \to X$, $i \in I$, is a semidynamical system, i.e.

$$\Pi_i(0, x) = x \quad \text{for every} \quad i \in I, \ x \in X \tag{3.1}$$

and

$$\Pi_i(s + t, x) = \Pi_i(s, \Pi_i(t, x)) \quad \text{for every} \quad s, t \in \mathbb{R}_+, \ i \in I \quad \text{and} \quad x \in X. \tag{3.2}$$

We assume that $\Pi_i : \mathbb{R}_+ \times X \to X$, $i \in I$ are continuous and that there exists $x_* \in X$ such that

$$\int_{\mathbb{R}_+} e^{-\lambda t} \|\Pi_i(t, x_*) - x_*\| \, dt < \infty \quad \text{for} \quad i \in I. \tag{3.3}$$

Moreover we assume that the functions p_{ij}, $i, j \in I$, satisfy the following condition

$$\sum_{j=1}^{N} |p_{ij}(x) - p_{ij}(y)| \leqslant w(\|x - y\|) \quad \text{for} \quad x, y \in X, \ i \in I, \tag{3.4}$$

where the function $w \in \Phi_0$ satisfies condition (2.4) and

$$\gamma = \inf \left\{ p_{ij}(x) : i, j \in I, \ x \in X \right\} > 0. \tag{3.5}$$

Further we assume that there exist constants $L \geqslant 1$ and $\alpha \in \mathbb{R}$ such that

$$\sum_{j=1}^{N} p_{ij}(y) \|\Pi_j(t, x) - \Pi_j(t, y)\| \leqslant L e^{\alpha t} \|x - y\| \quad \text{for} \quad x, y \in X, \ i \in I. \tag{3.6}$$

Finally we assume that there exists a constant $L_q > 0$ such that

$$\int_S \|q_s(x) - q_s(y)\| \kappa(ds) \leqslant L_q \|x - y\| \quad \text{for} \quad x, y \in X. \tag{3.7}$$

Let $(\tau_n)_{n \geqslant 0}$ and $(\zeta_n)_{n \geqslant 0}$ be sequences of random variables introduced in Section 1. Let $(\xi_n)_{n \geqslant 0}$ be the corresponding random dynamical system described in introduction. This process is not Markovian. We extend the process $(\xi_n)_{n \geqslant 0}$ in such a way that the new process becomes Markovian. In this purpose consider the space $X \times I$ endowed with the metric ρ given by

$$\rho\big((x, i), (y, j)\big) = \|x - y\| + \rho_0(i, j) \quad \text{for} \quad x, y \in X, \ i, j \in I, \tag{3.8}$$

where

$$\rho_0(i, j) = \begin{cases} c, & \text{if} \quad i \neq j, \\ 0, & \text{if} \quad i = j \end{cases} \tag{3.9}$$

with the constant c suitably choosen.

Let $(\eta_n)_{n\geqslant 0}$ be a sequence of random elements $\eta_n : \Omega \to I$, $n \in \mathbb{N}$, such that

$$\mathbb{P}\big(\eta_0 = i \mid \xi_0 = x\big) = p_i(x)$$

and

$$\mathbb{P}\big(\eta_n = j \mid \eta_{n-1} = i,\ \xi_n = x\big) = p_{ij}(x) \quad \text{for} \quad n = 1, 2, \ldots$$

Assume that $(\eta_n)_{n\geqslant 0}$ is independent upon $(\tau_n)_{n\geqslant 0}$ and that for every $n \in \mathbb{N}$ the random variables $\zeta_1, \ldots, \zeta_{n-1}, \eta_1, \ldots, \eta_{n-1}$ are also independent.

Given an initial random variable ξ_0 we consider the random process $(\xi_n)_{n\geqslant 0}$ defined by the formula

$$\xi_n = q_{\zeta_n}\big(\Pi_{\eta_{n-1}}(\Delta\tau_n, \xi_{n-1})\big) \quad \text{for} \quad n = 1, 2, \ldots$$

Now we consider a stochastic process $\big(\xi_n, \eta_n\big)_{n\geqslant 0}$. Clearly $\big(\xi_n, \eta_n\big) : \Omega \to X \times I$. It is easy to check that this process admits the Markov property.

Let μ_0 be the distribution of the initial random variable (ξ_0, η_0), i.e.

$$\mu_0(A) = \mathbb{P}\big((\xi_0, \eta_0) \in A\big) \quad \text{for} \quad A \in \mathcal{B}(X \times I).$$

For $n \in \mathbb{N}$ we denote by μ_n the distribution of (ξ_n, η_n), i.e.

$$\mu_n(A) = \mathbb{P}\big((\xi_n, \eta_n) \in A\big) \quad \text{for} \quad A \in \mathcal{B}(X \times I).$$

Proposition 3.1. There exists a Feller operator $P : \mathcal{M}(X \times I) \to \mathcal{M}(X \times I)$ such that

$$\mu_{n+1} = P\mu_n \quad \text{for every} \quad n \in \mathbb{N}. \tag{3.10}$$

Moreover, the operator P is given by the formula

$$P\mu(A) = \sum_{j=1}^{N} \int_{X\times I} \int_0^\infty \int_S \lambda e^{-\lambda t} \cdot 1_A\big(q_s\big(\Pi_j(t, x)\big), j\big)\, p_{ij}(x)\, \kappa(ds)\, dt\, \mu(dx\, di) \tag{3.11}$$

and its dual operator U by the formula

$$Uf(x,i)=\sum_{j=1}^{N}\int_{0}^{+\infty}\int_{S} f\big(q_s\big(\Pi_j(t,x)\big),j\big)p_{ij}(x)\lambda e^{-\lambda t}\kappa(ds)\,dt. \quad (3.12)$$

Proof. The proof is standard and we only give the main ideas of it. Let $(x,i)\in X\times I$ be given and let $f\in C(X\times I)$. Let $\mathbb{E}$ denote the mathematical expectation with respect to the probability $\mathbb{P}$. We have

$$\mathbb{E}\left(f\left(\xi_{n+1},\eta_{n+1}\right)\right)=\int_{X\times I} f(x,i)\mu_{n+1}(dx\,di)=<f,\mu_{n+1}>. \quad (3.13)$$

Since $(\zeta_n)_{n\geqslant 0}$ are independent on $(\tau_n)_{n\geqslant 0}$ we have

$$\mathbb{E}\big(f\left(\xi_{n+1},\eta_{n+1}\right)\big)=\sum_{i=1}^{N}\int_{\Omega} f\big(q_{\zeta_{n+1}}\big(\Pi_{\eta_{n+1}}(\Delta\tau_{n+1},\xi_n)\big),\eta_{n+1}\big)1_{\{\omega:\eta_n=i\}}d\mathbb{P}.$$

On the other hand, since $\Delta\tau_{n+1}$ and ζ_{n+1} are independent on η_{n+1}, η_n and ξ_n we obtain

$$\mathbb{E}\big(f\left(\xi_{n+1},\eta_{n+1}\right)\big)=\int_{X\times I}\sum_{j=1}^{N}\int_{0}^{\infty}\int_{S} f\big(q_s\big(\Pi_j(t,x)\big),j\big)p_{ij}(x)\lambda e^{-\lambda t} \times\kappa(ds)\,dt\,\mu_n(dx\,di).$$

Now, using notation (3.12) we can write

$$\mathbb{E}\big(f\left(\xi_{n+1},\eta_{n+1}\right)\big)=\int_{X\times I} Uf(x,i)\mu_n(dx\,di)=<Uf,\mu_n>. \quad (3.14)$$

Further, simple calculation shows that:

(i) $Uf\geqslant 0$ for $f\in B(X\times I)$ and $f\geqslant 0$;

(ii) $U1_{X\times I}=1_{X\times I}$;

(iii) $Uf_n\downarrow 0$ for $f_n\in B(X\times I)$ and $f_n\downarrow 0$;

(iv) $Uf\in C(X\times I)$ for $f\in C(X\times I)$.

Therefore by ref. 18 there exists a Feller operator P such that

$$<Uf, \mu_n> = < f, P\mu_n> \quad \text{for} \quad f \in B(X \times I) \quad \text{and} \quad \mu \in \mathcal{M}_1(X \times I). \tag{3.15}$$

By (3.13), (3.14) and (3.15) for arbitrary $A \in \mathcal{B}(X \times I)$ we have

$$\mu_{n+1}(A) = <1_A, \mu_{n+1}> = <U1_A, \mu_n> = <1_A, P\mu_n> = P\mu_n(A),$$

which proves condition (3.10). Moreover, by (3.15) and (3.12) we have

$$\begin{aligned} P\mu(A) = & <1_A, P\mu> = <U1_A, \mu> \\ & = \sum_{j=1}^{N} \int_{X \times I} \int_0^\infty \int_S \lambda e^{-\lambda t} 1_A\big(q_s\big(\Pi_j(t,x), j\big)\big) p_{ij}(x)\, \kappa(ds)\, dt\, \mu(dx\, di), \end{aligned}$$

which completes the proof. ∎

4. LEMMAS

Lemma 4.1. Assume that the system (Π, q, p) satisfies conditions (3.1)–(3.7). Moreover assume that

$$LL_q + \frac{\alpha}{\lambda} < 1, \tag{4.1}$$

where L, L_q and α are constants appearing in conditions (3.6), (3.7) and λ is the intesivity of the Poisson process which governs the increment $\Delta\tau_n$ of random variables $(\tau_n)_{n\geqslant 0}$. Then the operator P given by (3.11) is essentially nonexpansive.

Proof. Let $w \in \Phi_0$ be given by condition (3.4). Let $\varphi \in \Phi$ be such that inequality (2.5) holds with

$$a = \frac{\lambda L L_q}{\lambda - \alpha}. \tag{4.2}$$

Since $a < 1$ (see (4.1)), φ exists by virtue of Proposition 2.3.

Let $c \in \mathbb{R}_+$ be such that

$$\varphi(c) > 2 \tag{4.3}$$

and let ρ be given by (3.8) and (3.9).

Fix $f \in \mathcal{F}_\varphi$. To complete the proof it is enough to show that

$$|Uf(x,i) - Uf(y,j)| \leqslant \varphi(\varrho((x,i),(y,j))), \quad \text{for} \quad (x,i),(y,j) \in X \times I, \tag{4.4}$$

where the operator U is given by (3.12). From (3.9) and (4.3) it follows that (4.4) holds if $i \neq j$.

By (3.12) and the last inequality we have

$$\begin{aligned}
&|Uf(x,i) - Uf(y,i)| \\
&\leqslant \sum_{j=1}^{N} \int_0^{+\infty} \int_S \lambda e^{-\lambda t} | f\big(q_s\big(\Pi_j(t,x)\big), j\big) p_{ij}(x) \\
&\quad - f\big(q_s\big(\Pi_j(t,y)\big), j\big) p_{ij}(y) | \kappa(ds)dt \\
&\leqslant \sum_{j=1}^{N} |p_{ij}(x) - p_{ij}(y)| \\
&\quad + \sum_{j=1}^{N} \int_0^{+\infty} \int_S \lambda e^{-\lambda t} p_{ij}(y) \varphi\left(\|q_s\big(\Pi_j(t,x)) - q_s(\Pi_j(t,y)\big)\| \right) \kappa(ds)dt
\end{aligned}$$

and using in turn (3.4), (2.4), the Jensen inequality and finally (3.7), (3.6) and (2.5) we obtain

$$\begin{aligned}
&|Uf(x,i) - Uf(y,i)| \\
&\leqslant \omega(\|x-y\|) + \int_0^{+\infty} \int_S \lambda e^{-\lambda t} \varphi\left(\sum_{j=1}^{N} p_{ij}(y) \|q_s\big(\Pi_j(t,x)\big) - q_s\big(\Pi_j(t,y)\big)\| \right) \kappa(ds)dt \\
&\leqslant \omega(\|x-y\|) + \varphi\left(\sum_{j=1}^{N} \int_0^{+\infty} \int_S \lambda e^{-\lambda t} p_{ij}(y) \|q_s\big(\Pi_j(t,x)\big) - q_s\big(\Pi_j(t,y)\big)\| \kappa(ds)dt \right) \\
&\leqslant \omega(\|x-y\|) + \varphi\left(\int_0^{+\infty} \lambda e^{-\lambda t} \sum_{j=1}^{N} p_{ij}(y) L_q \|\Pi_j(t,x) - \Pi_j(t,y)\| dt \right) \\
&\leqslant \omega(\|x-y\|) + \varphi\left(\lambda L L_q \|x-y\| \int_0^{+\infty} e^{(\alpha-\lambda)t} dt \right) \\
&= \omega(\|x-y\|) + \varphi(a\|x-y\|) \leqslant \varphi(\|x-y\|),
\end{aligned}$$

which completes the proof. ∎

Lemma 4.2. Suppose that hypotheses of Lemma 4.1 hold. Then there exists a bounded set $A \subset X \times I$ such that

$$\inf_{\mu \in \mathcal{M}(X \times I)} \liminf_{n \to \infty} P^n \mu(A) > 0.$$

Proof. Put

$$V(x, i) = \|x\| \quad \text{for} \quad (x, i) \in X \times I.$$

Claim. There exist $a, b \in \mathbb{R}_+$, $a < 1$, such that

$$UV(x, i) \leqslant aV(x, i) + b \quad \text{for} \quad (x, i) \in X \times I. \tag{4.5}$$

Indeed, using (3.12) and the definition of V we have

$$\begin{aligned} UV(x, i) &= \sum_{j=1}^{N} \int_0^{+\infty} \int_S \|q_s(\Pi_j(t, x))\| \lambda e^{-\lambda t} p_{ij}(x) \kappa(ds) dt \\ &\leqslant \sum_{j=1}^{N} \int_0^{+\infty} \int_S \|q_s(\Pi_j(t, x)) - q_s(\Pi_j(t, x_*))\| \lambda e^{-\lambda t} \\ &\quad \times p_{ij}(x) \kappa(ds) dt \\ &\quad + \sum_{j=1}^{N} \int_0^{+\infty} \int_S \|q_s(\Pi_j(t, x_*))\| \lambda e^{-\lambda t} p_{ik}(x) \kappa(ds) dt, \end{aligned}$$

where x_* is given by (3.3).

Further, using (3.7) and then (3.6) we obtain

$$\begin{aligned} UV(x, i) \leqslant &\sum_{j=1}^{N} \int_0^{+\infty} \lambda e^{-\lambda t} \left[\int_S \|q_s\big(\Pi_j(t, x)\big) - q_s\big(\Pi_j(t, x_*)\big)\| \kappa(ds) \right] p_{ij}(x) dt \\ &+ \sum_{j=1}^{N} \int_0^{+\infty} \lambda e^{-\lambda t} \left[\int_S \|q_s\big(\Pi_j(t, x_*)\big) - q_s(x_*)\| \kappa(ds) \right] p_{ij}(x) dt \\ &+ \int_0^{+\infty} \lambda e^{-\lambda t} \left[\int_S \|q_s(x_*)\| \kappa(ds) \right] dt \end{aligned}$$

$$\begin{aligned}
&\leqslant L_q \int_0^{+\infty} \lambda e^{-\lambda t} \sum_{j=1}^{N} p_{ij}(x) \|\Pi_j(t,x) - \Pi_j(t,x_*)\| dt \\
&+ L_q \sum_{j=1}^{N} \int_0^{+\infty} \lambda e^{-\lambda t} p_{ij}(x) \|\Pi_j(t,x_*) - x_*\| dt + \int_S \|q_s(x_*)\| \kappa(ds) \\
&\leqslant \lambda L L_q \int_0^{+\infty} e^{(\alpha-\lambda)t} dt \cdot \|x - x_*\| + \widetilde{b} = a\|x - x_*\| + \widetilde{b} \\
&\leqslant a\|x\| + b,
\end{aligned}$$

where a is given by (4.2),

$$\widetilde{b} = \lambda L_q \sum_{j=1}^{N} \int_0^{+\infty} e^{-\lambda t} p_{ij}(x) \|\Pi_j(t,x_*) - x_*\| dt + \int_S \|q_s(x_*)\| \kappa(ds)$$

and

$$b = \widetilde{b} + a\|x_*\|.$$

From (3.3) and the fact that $q(\cdot, x_*)$ is a bounded function, it follows that b is finite. Since $a < 1$ the proof of the Claim is complete.

From (4.5) it follows that

$$U^n V(x,i) \leqslant a^n V(x,i) + \frac{b}{1-a} \quad \text{for} \quad (x,i) \in X \times I.$$

Let $\mu \in \mathcal{M}_1(X \times I)$ be given and let $K \subset X \times I$ be a compact set such that $\mu(K) \geqslant 1/2$. Define $\bar{\mu}(B) = \mu(B \cap K)$ for $B \in \mathcal{B}(X \times I)$. Further let

$$A = \{(x,i) \in X \times I : V(x,i) \leqslant d\},$$

where $d = 4b/(1-a)$. From the Chebyshev inequality it follows that

$$P^n \mu(A) \geqslant P^n \bar{\mu}(A) \geqslant \frac{1}{2} - \frac{1}{d} \int_{X \times I} V \, d P^n \bar{\mu}$$

and consequently

$$\begin{aligned}
P^n \mu(A) &\geqslant \frac{1}{2} - \frac{1}{d} \left(a^n \int_{X \times I} V \, d\bar{\mu} + \frac{b}{1-a} \right) \\
&\geqslant \frac{1}{4} - \frac{a^n}{d} \int_{X \times I} V \, d\bar{\mu}.
\end{aligned}$$

Since the support of $\bar{\mu}$ is compact, the last integral is bounded and the statement of Lemma 4.2 follows. ∎

Lemma 4.3. Under the assumptions of Lemma 4.1 the operator P given by (3.11) is semi-concentrating.

Proof. Define

$$\mathcal{E}(P)=\Big\{\varepsilon>0:\ \inf_{\mu\in\mathcal{M}_1}\ \liminf_{n\to\infty} P^n\mu(A)>0 \text{ for some } A\in\mathcal{C}_\varepsilon(X\times I)\Big\}.$$

To complete the proof it is sufficient to show that $\inf\mathcal{E}(P)=0$. Suppose, on the contrary, that $\widetilde{\varepsilon}=\inf\mathcal{E}(P)>0$. Let α be given by condition (3.6). We consider two cases: $\alpha<0$ and $\alpha\geqslant 0$.

Case I. Suppose first that $\alpha<0$. By Lemma 4.2 there exist $x_0\in X$ and $r>0$ such that

$$\inf_{\mu\in\mathcal{M}_1(X\times I)}\ \liminf_{n\to\infty} P^n\mu\big(B(x_0,r)\times I\big)>0. \tag{4.6}$$

Fix $t_*>0$ such that

$$\varepsilon=3rLL_q e^{\alpha t_*}<\widetilde{\varepsilon} \tag{4.7}$$

and set

$$C_\varepsilon=\bigcup_{j=1}^{N}\ \bigcup_{t\in[t^*,2t^*]}\ \bigcup_{s\in S}\left(B\big(q_s(\Pi_j(t,x_0)),\ \frac{2\varepsilon}{3}\big)\times I\right).$$

Observe that $C_\varepsilon\in\mathcal{C}_\varepsilon$.

According to (3.11), for arbitrary $\mu\in\mathcal{M}_1(X\times I)$ we have

$$\begin{aligned} &P^{n+1}\mu(C_\varepsilon)\\ &\quad=\sum_{j=1}^{N}\int_{X\times I}\int_0^{+\infty}\int_S 1_{C_\varepsilon}(q_s(\Pi_j(t,x)),\,j)\lambda e^{-\lambda t}\\ &\qquad\times p_{ij}(x)\kappa(ds)dt\,P^n\mu(dx\,di). \end{aligned}\tag{4.8}$$

Let $x\in B(x_0,r)$ and $t>t_*$ be fixed. Since $\sum_{j=1}^{N}p_{ij}(x)=1$, from (3.6) it follows that there is $j\in I$ (depending on x and t) such that

$$\|\Pi_j(t,x)-\Pi_j(t,x_0)\|\leqslant Le^{\alpha t}\|x-x_0\|. \tag{4.9}$$

Further, by (3.7)

$$\int_S \|q_s\big(\Pi_j(t,x)\big) - q_s\big(\Pi_j(t,x_0)\big)\|\kappa(ds) \leqslant L_q \|\Pi_j(t,x) - \Pi_j(t,x_0)\|.$$

From the last inequality it follows that

$$\kappa\big(S(x,t;j)\big) \geqslant 1/2,$$

where

$$\begin{aligned} S(x,t;j) = \big\{s \in S : \|q_s\big(\Pi_j(t,x)\big) - q_s\big(\Pi_j(t,x_0)\big)\| \\ \leqslant 2L_q \|\Pi_j(t,x) - \Pi_j(t,x_0)\|\big\}. \end{aligned}$$

For $s \in S(x,t;j)$ with $x \in B(x_0, r)$ and $t > t_*$ by (4.7) and (4.9) we have

$$\begin{aligned} \|q_s\big(\Pi_j(t,x)\big) - q_s\big(\Pi_j(t,x_0)\big)\| &\leqslant 2L_q \|\Pi_j(t,x) - \Pi_j(t,x_0)\| \\ &\leqslant 2LL_q e^{\alpha t} \|x - x_0\| \leqslant 2\varepsilon/3. \end{aligned}$$

This means that for every $x \in B(x_0, r)$ and $t > t_*$ there is $j \in I$ such that

$$\big(q_s(\Pi_j(t,x)), j\big) \in C_\varepsilon$$

and consequently

$$\sum_{j=1}^{N} 1_{C_\varepsilon}(q_s(\Pi_j(t,x)), j) \geqslant 1.$$

From (4.7) it follows that

$$\begin{aligned} P^{n+1}\mu(C_\varepsilon) &\geqslant \int_{B(x_0,r)\times I} \int_{t_*}^{2t_*} \int_S \lambda e^{-\lambda t} p_{ij}(x) \kappa(ds) dt \, P^n \mu(dx\, di) \\ &\geqslant \frac{\gamma}{2} e^{-\lambda t_*}(1 - e^{-\lambda t_*}) \cdot P^n \mu\big(B(x_0, r) \times I\big), \end{aligned}$$

where γ is given by (3.5). From (4.6) and the last inequality it follows that

$$\inf_{\mu \in \mathcal{M}_1(X\times I)} \liminf_{n\to\infty} P^n \mu\big(C_\varepsilon\big) > 0,$$

which contradicts to the fact that $\widetilde{\varepsilon} = \inf \mathcal{E}(P)$. Consequently $\widetilde{\varepsilon} = 0$.

Case II. Suppose now that $\alpha \geqslant 0$. Then by (4.1) we have $LL_q < 1$. Choose $\sigma > 0$ and $t_* > 0$ such that

$$(1+\sigma)LL_q e^{\alpha t_*} < 1.$$

Finally choose $\varepsilon_0 > \widetilde{\varepsilon}$ such that

$$\varepsilon = (1+\sigma)LL_q e^{\alpha t_*} \varepsilon_0 < \widetilde{\varepsilon}.$$

By the definition of $\mathcal{E}(P)$ there is $A \in \mathcal{C}_{\varepsilon_0}$ such that

$$\beta = \inf_{\mu \in \mathcal{M}_1(X \times I)} \liminf_{n \to \infty} P^n \mu\big(A\big) > 0. \tag{4.10}$$

We may assume that

$$A = \bigcup_{k=1}^{m} \big(B(x_k, \varepsilon_0) \times I\big). \tag{4.11}$$

Now we define

$$C_\varepsilon = \bigcup_{j=1}^{N} \bigcup_{t \in [0,t_*]} \bigcup_{s \in S} \bigcup_{k=1}^{m} \left(B\big(q_s(\Pi_j(t, x_k)), \frac{2\varepsilon}{3}\big) \times I \right).$$

Let $\mu \in \mathcal{M}_1(X \times I)$ be arbitrary. From (4.10) and (4.11) it follows that there is $k_n \in \{1, \ldots, m\}$ such that

$$P^n \mu\big(B(x_{k_n}, \varepsilon_0) \times I\big) \geqslant \frac{\beta}{m}. \tag{4.12}$$

Further, from (3.6) it follows that for every $x_{k_n} \in X$ and $t \in \mathbb{R}_+$ there is $j \in I$ (depending on x_{k_n} and t) such that inequality (4.9) with x_{k_n} in place of x_0, holds. Simple calculation shows that

$$\kappa\big(S(x, t, j, \sigma)\big) \geqslant \frac{\sigma}{1+\sigma},$$

where

$$\begin{aligned} S(x, t; j, \sigma) = \big\{ & s \in S : \|q_s\big(\Pi_j(t, x)\big) - q_s\big(\Pi_j(t, x_{k_n})\big)\| \\ & \leqslant (1+\sigma)L_q \|\Pi_j(t, x) - \Pi_j(t, x_{k_n})\| \big\}. \end{aligned}$$

Argument similar to that of case I gives

$$P^{n+1}\mu(C_\varepsilon) \geqslant \frac{\gamma\sigma}{1+\sigma} \int_0^{t_*} \lambda e^{-\lambda t} dt \cdot P^n \mu\big(B(x_{k_n}, \varepsilon_0) \times I\big).$$

From the last inequality and (4.12) it follows immediately that

$$\liminf_{n\to\infty} P^n \mu\big(C_\varepsilon\big) \geqslant \frac{\gamma\sigma\beta}{(1+\sigma)m} \big(1 - e^{-\lambda t_*}\big).$$

Since $\mu \in \mathcal{M}_1(X \times I)$ was arbitrary and $\varepsilon < \widetilde{\varepsilon}$, this contradicts to the fact that $\widetilde{\varepsilon} = \inf \mathcal{E}(P)$. Consequently $\widetilde{\varepsilon} = 0$ and the proof is complete. ∎

5. MAIN RESULTS

Theorem 5.1. Under the hypotheses of Lemma 4.1 the operator P defined by (3.11) admits an invariant measure.

Proof. By Lemma 4.1 and 4.2 the operator P is essentially nonexpansive and semi-concentrating. Thus, the statement of Theorem 5.1 follows from Proposition 2.2. ∎

Theorem 5.2. Under the hypotheses of Lemma 4.1 the operator P defined by (3.11) is asymptotically stable.

Proof. By Theorem 5.1 the operator P admits an invariant measure. By virtue of Proposition 2.1 it is sufficient to show that for $\varepsilon > 0$ there exists $\theta > 0$ such that for every two measures $\mu_1, \mu_2 \in \mathcal{M}_1(X \times I)$ there exist a Borel measurable set $A \subset X \times I$ with $\operatorname{diam} A < \varepsilon$ and an integer $\tilde{n}$ such that

$$P^{\tilde{n}} \mu_k(A) \geqslant \theta \quad \text{for} \quad k = 1, 2.$$

Since by Proposition 2.2 the set $\mathcal{L}(\mathcal{M}_1)$ is tight, there exists a compact set $K \subset X \times I$ such that

$$\mu(K) \geqslant \frac{4}{5} \quad \text{for every} \quad \mu \in \mathcal{L}(\mathcal{M}_1).$$

Let α be given by condition (3.6). Analogously as in the proof of Lemma 4.3 we consider two cases: $\alpha < 0$ and $\alpha \geqslant 0$.

Case I. Suppose first that $\alpha < 0$. Let $\varepsilon > 0$ be fixed. Choose $t_* \in \mathbb{R}_+$ such that

$$LL_q e^{\alpha t_*} \operatorname{diam} K < \frac{\varepsilon}{4}, \tag{5.1}$$

where L, L_q are given by conditions (3.6) and (3.7), respectively.

Define

$$K_X = \left\{ x \in X : (x, i) \in K \text{ for some } i \in I \right\}$$

and

$$K_X^* = \bigcup_{j=1}^{N} \Pi_j(t_*, K_X).$$

Clearly K_X and K_X^* are compact subsets of X. For $\tilde{s} \in S$ define

$$V(\tilde{s}) = \left\{ s \in S : \|q_s(y) - q_{\tilde{s}}(y)\| < \frac{\varepsilon}{12} \text{ for every } y \in K_X^* \right\}. \tag{5.2}$$

Since $V(\tilde{s})$ is an open neighborhood of $\tilde{s}$ and S is a compact space, there exists a finite set $\{s_1, \dots, s_{\overline{m}}\}$ such that $S = \cup_{j=1}^{\overline{m}} V(s_j)$. Set $V_j = V(s_j)$, $j = 1, \dots, \overline{m}$, and define

$$\vartheta = \inf_{j \in J} \kappa(V_j), \tag{5.3}$$

where

$$J = \left\{ j \in \{1, \dots, \overline{m}\} : \kappa(V_j) > 0 \right\}. \tag{5.4}$$

Obviously $\vartheta > 0$ and $\sum_{j \in J} \kappa(V_j) \geqslant 1$.

Now for $x \in K_X$ we set

$$O(x) = \left\{ z \in K_X : \|q_s\big(\Pi_i(t_*, z)\big) - q_s\big(\Pi_i(t_*, x)\big)\| < \frac{\varepsilon}{12} \text{ for } s \in S,\ i \in I \right\}, \tag{5.5}$$

where t_* is given by condition (5.1). Let $x_1, \dots, x_{m_0} \in K_X$ be such that $K \subset G$, where

$$G = \bigcup_{l=1}^{m_0} \big(O(x_l) \times I\big). \tag{5.6}$$

Note that G is an open subset of $X \times I$. By compactness and continuity there exists $\hat{t} > t_*$ such that

$$\|q_s\big(\Pi_i(t,x)\big) - q_s\big(\Pi_i(t_*,x)\big)\| < \frac{\varepsilon}{12}, \tag{5.7}$$

for every $i \in I$, $s \in S$, $x \in K_X$ and $t \in [t_*, \hat{t}]$.

Let $\mu_1, \ \mu_2 \in \mathcal{M}_1(X \times I)$ be arbitrary. Set $\mu = (\mu_1 + \mu_2)/2$. Since $\mathcal{L}(\mu) \neq \emptyset$ (see Proposition 2.2) there exists a sequence $(n_k)_{k \geqslant 1}$ and a measure $\nu \in \mathcal{L}(\mu)$ such that $P^{n_k}\mu \to \nu$ (weakly). Since $\nu(G) \geqslant 4/5$ by the Alexandrov Theorem there exists $n_0 \in \mathbb{N}$ such that

$$P^{n_0}\mu(G) \geqslant \frac{3}{4}.$$

Consequently,

$$P^{n_0}\mu_k(G) \geqslant \frac{1}{2} \quad \text{for} \quad k = 1, 2.$$

Therefore there exists $l_1, l_2 \in \{1, \dots, m_0\}$ and $i_1, i_2 \in I$ such that

$$P^{n_0}\mu_k(\mathcal{V}_k) \geqslant \frac{1}{2m_0 N} \quad \text{for} \quad k = 1, 2, \tag{5.8}$$

where

$$\mathcal{V}_k = O(x_{l_k}) \times \{i_k\}, \quad k = 1, 2.$$

From condition (3.6) it follows that there is an $i_0 \in I$ such that

$$\|y_1 - y_2\| \leqslant L e^{\alpha t_*} \|x_{l_1} - x_{l_2}\|, \tag{5.9}$$

where

$$y_1 = \Pi_{i_0}(t_*, x_{l_1}), \quad y_2 = \Pi_{i_0}(t_*, x_{l_2}).$$

Moreover, from condition (3.7) it follows that there exists $S_0 \subset S$ with $\kappa(S_0) > 0$ such that

$$\|q_s(y_1) - q_s(y_2)\| \leqslant L_q \|y_1 - y_2\| \quad \text{for every} \quad s \in S_0. \tag{5.10}$$

Since $\kappa(S_0) > 0$ there exists an $j_0 \in J$ such that $S_0 \cap V_0 \neq \emptyset$, where $V_0 = V(s_{j_0})$. Fix $\tilde{s} \in S_0 \cap V_0$. By (5.2), (5.10), (5.9) and (5.1) we have

$$\begin{aligned}\|q_{s_{j_0}}(y_2) - q_{s_{j_0}}(y_1)\| &\leqslant \|q_{s_{j_0}}(y_2) - q_{\tilde{s}}(y_2)\| + \|q_{\tilde{s}}(y_2) - q_{\tilde{s}}(y_1)\| \\ &+ \|q_{\tilde{s}}(y_1) - q_{s_{j_0}}(y_1)\| \leqslant \frac{\varepsilon}{12} + L_q\|y_1 - y_2\| + \frac{\varepsilon}{12} \leqslant \frac{\varepsilon}{6} + \frac{\varepsilon}{4} < \frac{\varepsilon}{2}.\end{aligned}$$

Define

$$A = \Big(B\big(q_{s_{j_0}}(y_1), \frac{\varepsilon}{4}\big) \cup B\big(q_{s_{j_0}}(y_2), \frac{\varepsilon}{4}\big)\Big) \times \{i_0\}.$$

Obviously $\operatorname{diam} A < \varepsilon$.

For $s \in V_0$, $x \in O(x_{l_1})$ and $t \in [t_*, \hat{t}]$, using (5.7), (5.5) and (5.2) we have

$$\begin{aligned}\|q_s\big(\Pi_{i_0}(t,x)\big) - q_{s_{j_0}}(y_1)\| &\leqslant \|q_s\big(\Pi_{i_0}(t,x)\big) - q_s\big(\Pi_{i_0}(t_*,x)\big)\| \\ &+ \|q_s\big(\Pi_{i_0}(t_*,x)\big) - q_s\big(\Pi_{i_0}(t_*,x_{l_1})\big)\| + \|q_s(y_1) - q_{s_{j_0}}(y_1)\| \\ &< \frac{\varepsilon}{12} + \frac{\varepsilon}{12} + \frac{\varepsilon}{12} = \frac{\varepsilon}{4}.\end{aligned}$$

This means that

$$\big(q_s\big(\Pi_{i_0}(t,x)\big), i_0\big) \in A \quad \text{for} \quad s \in V_0,\ x \in O(x_{l_1}), \quad t \in [t_*, \hat{t}]. \tag{5.11}$$

By (3.11), (5.11), (3.5), (5.3) and (5.8) we have

$$\begin{aligned}&P^{n_0+1}\mu_1(A) \\ &= \sum_{j=1}^{N} \int_{X\times I} \int_0^\infty \int_S 1_A\big(q_s\big(\Pi_j(t,x)\big), j\big)\lambda e^{-\lambda t} p_{ij}(x)\,\kappa(ds)\,dt\,P^{n_0}\mu_1(dx\,di) \\ &\geqslant \int_{\mathcal{V}_1} \int_{t_*}^{\hat{t}} \int_{V_0} 1_A\big(q_s\big(\Pi_{i_0}(t,x)\big), i_0)\big)\lambda e^{-\lambda t} p_{ii_0}(x)\,\kappa(ds)\,dt\,P^{n_0}\mu_1(dx\,di) \\ &\geqslant \gamma\kappa(V_0)P^{n_0}\mu_1(\mathcal{V}_1)\int_{t_*}^{\hat{t}} \lambda e^{-\lambda t}dt \geqslant \frac{\gamma\vartheta}{2m_0N}\big(e^{-\lambda t_*} - e^{-\lambda\hat{t}}\big).\end{aligned}$$

The same argument shows that the last inequality holds for μ_2. Since the constant $\theta = \gamma\vartheta(e^{-\lambda t_*} - e^{-\lambda\hat{t}})/(2m_0N)$ does not depend on μ_1 and μ_2, the proof of the first case is complete.

Case II. Suppose now that $\alpha \geqslant 0$. We introduce some further notation. Namely, for $\mathbf{s} \in S^n$, $\mathbf{i} \in I^n$ and $\mathbf{t} \in \mathbb{R}^n_+$ (i.e. $\mathbf{s} = (s_1, \ldots, s_n)$, $\mathbf{t} = (t_1, \ldots, t_n)$ and $\mathbf{i} = (i_1, \ldots, i_n)$) we set

$$\begin{aligned} \mathbf{q_s} &= q_{s_n} \circ \cdots \circ q_{s_1}, \\ (\mathbf{q_s} \circ \Pi_\mathbf{i})(\mathbf{t}, x) &= q_{s_n}\big(\Pi_{i_n}\big(t_n, q_{s_{n-1}}(\Pi_{i_{n-1}}(t_{n-1}, \ldots, \Pi_{i_1}(t_1, x)))\big)\big), \\ \mathbf{dt} &= dt_1 \cdots dt_n, \\ \mathbf{ds} &= ds_1 \cdots ds_n. \end{aligned}$$

Moreover κ^n stands for the measure on S^n generated by κ (i.e. $\kappa^n = \underbrace{\kappa \otimes \cdots \otimes \kappa}_{n-\text{times}}$).

Since $\alpha \geqslant 0$ condition (4.1) implies that $L_q < 1$. Let $n \in \mathbb{N}$ be such that

$$L_q^n \cdot \operatorname{diam} K < \frac{\varepsilon}{12}. \tag{5.12}$$

By continuity and compactness there exists $\sigma > 0$ such that

$$\|\big(\mathbf{q_s} \circ \Pi_\mathbf{i}\big)(\mathbf{t}, x) - \mathbf{q_s}(x)\| < \frac{\varepsilon}{12} \tag{5.13}$$

for every $\mathbf{i} \in I^n$, $\mathbf{s} \in S^n$, $\mathbf{t} \in [0, \sigma]^n$ and $x \in K_X$.

Given $\tilde{\mathbf{s}} \in S^n$ we define

$$\mathbf{V}(\tilde{\mathbf{s}}) = \left\{\mathbf{s} \in S^n : \|\mathbf{q_s}(x) - \mathbf{q}_{\tilde{\mathbf{s}}}(x)\| < \frac{\varepsilon}{24} \quad \text{for every} \quad x \in K_X\right\}. \tag{5.14}$$

Clearly $\mathbf{V}(\tilde{\mathbf{s}})$ is an open neighborhood of $\tilde{\mathbf{s}}$. Since S^n is compact, there exists a finite family $\mathbf{V}_j = \mathbf{V}(\mathbf{s}_j)$, $j = 1, \ldots, \overline{m}$, such that $S^n = \cup_{j=1}^{\overline{m}} \mathbf{V}_j$. Set

$$J = \big\{j \in \{1, \ldots, \overline{m}\} : \kappa^n(\mathbf{V}_j) > 0\big\}$$

and

$$\vartheta = \min_{j \in J} \kappa^n(\mathbf{V}_j). \tag{5.15}$$

Clearly $\vartheta > 0$. Given $x \in X$ we define

$$O(x)=\left\{z\in X:\|\mathbf{q}_{\mathbf{s}_j}(x)-\mathbf{q}_{\mathbf{s}_j}(z)\|<\frac{\varepsilon}{12}\quad\text{for}\quad j\in J\right\}. \tag{5.16}$$

Clearly $O(x)$ is an open neighborhood of x. Let $x_1,\dots,x_{m_0}\in K_X$ be such that $K\subset G$ where G is given by (5.6).

Let $\mu_1,\mu_2\in\mathcal{M}_1(X\times I)$. Let x_{l_1}, x_{l_2}, i_1, i_2 and $O(x_{l_1})$, $O(x_{l_2})$ be defined as in Case 1. From condition (3.7) it follows that there exists $\mathbf{S}_0\subset S^n$ such that $\kappa^n(\mathbf{S}_0)>0$ and

$$\|\mathbf{q}_{\mathbf{s}}(x_{l_1})-\mathbf{q}_{\mathbf{s}}(x_{l_2})\|\leqslant L_q^n\|x_{l_1}-x_{l_2}\|\quad\text{for every}\quad \mathbf{s}\in\mathbf{S}_0. \tag{5.17}$$

Since $\mathbf{S}_0$ is of positive measure, there exists $j_0\in J$ such that $\mathbf{S}_0\cap\mathbf{V}_0\neq\emptyset$, where $\mathbf{V}_0=\mathbf{V}(\mathbf{s}_{j_0})$. Choose $\mathbf{s}_0\in\mathbf{S}_0\cap\mathbf{V}_0$ and define

$$A=\Big(B\big(\mathbf{q}_{\mathbf{s}_0}(x_{l_1}),\frac{\varepsilon}{4}\big)\cup B\big(\mathbf{q}_{\mathbf{s}_0}(x_{l_2}),\frac{\varepsilon}{4}\big)\Big)\times\{i_0\}.$$

From (5.12) and (5.17) it follows that $\operatorname{diam}A<\varepsilon$.

For $\mathbf{s}\in\mathbf{V}_0$, $\mathbf{i}\in I^n$, $\mathbf{t}\in[0,\sigma]^n$ and $x\in O(x_{l_k})$ by virtue of (5.13)–(5.16) we have

$$\begin{aligned}\|\big(\mathbf{q}_{\mathbf{s}}\circ\Pi_{\mathbf{i}}\big)(\mathbf{t},x)-\mathbf{q}_{\mathbf{s}_0}(x_{l_k})\| \leqslant{}& \|\big(\mathbf{q}_{\mathbf{s}}\circ\Pi_{\mathbf{i}}\big)(\mathbf{t},x)-\mathbf{q}_{\mathbf{s}}(x)\|+\|\mathbf{q}_{\mathbf{s}}(x)-\mathbf{q}_{\mathbf{s}_{j_0}}(x)\|\\ &+\|\mathbf{q}_{\mathbf{s}_{j_0}}(x)-\mathbf{q}_{\mathbf{s}_{j_0}}(x_{l_k})\|+\|\mathbf{q}_{\mathbf{s}_{j_0}}(x_{l_k})-\mathbf{q}_{\mathbf{s}_0}(x_{l_k})\|<\varepsilon.\end{aligned}$$

This means that

$$\big((\mathbf{q}_{\mathbf{s}}\circ\Pi_{\mathbf{i}})(\mathbf{t},x),i_0\big)\in A\quad\text{for}\quad x\in O(x_{l_k}),\ \mathbf{s}\in\mathbf{V}_0\ \text{and}\ \mathbf{t}\in[0,\sigma]^n. \tag{5.18}$$

By (3.11), (5.18), (3.5), (5.15) and (5.8) we have

$$\begin{aligned}&P^{n_0+n}\mu_k(A)\\ &\quad=\sum_{\mathbf{i}\in I^n}\int_{X\times I}\int_{\mathbb{R}_+^n}\int_{S^n}1_A\big(\big((\mathbf{q}_s\circ\Pi_{\mathbf{i}})(\mathbf{t},x)\big),i_n\big)\lambda^n e^{-\lambda(t_1+\cdots+t_n)}\\ &\qquad\times p_{ii_1}(x)p_{i_1i_2}(q_{s_1}(t,x))\\ &\quad\cdots\cdot p_{i_{n-1}i_n}(q_{s_{n-1}}(\Pi_{i_{n-1}}(t_{n-1},\dots,\Pi_{i_1}(t_1,x))))\big)\,\kappa^n(\mathbf{ds})\,\mathbf{dt}\,P^{n_0}\mu_k(dx\,di)\\ &\quad\geqslant\int_{\mathcal{V}_1}\int_{[0,\sigma]^n}\int_{\mathbf{V}_0}\gamma^n\lambda^n e^{-\lambda(t_1+\cdots+t_n)}\,\kappa^n(\mathbf{ds})\,\mathbf{dt}\,P^{n_0}\mu_k(dx\,di)\\ &\quad\geqslant\gamma^n\left(\int_0^\sigma\lambda e^{-\lambda t}dt\right)^n\kappa^n(\mathbf{V}_0)P^{n_0}\mu_1(\mathcal{V}_1)\\ &\quad\geqslant\frac{\gamma\vartheta}{2m_0N}\big(1-e^{-\lambda\sigma}\big)^n\qquad\text{for}\qquad k=1,2.\end{aligned}$$

This completes the proof. ∎

Theorem 5.3. Assume that hypotheses of Theorem 5.1 hold. Then the sequence $(\widetilde{\mu}_n)$ of distributions corresponding to process $(\xi_n)_{n\geqslant 0}$ converges weakly to some distribution $\widetilde{\mu}_* \in \mathcal{M}_1(X)$.

Proof. Let $\widetilde{\mu}_n$ be the distribution of ξ_n. For arbitrary Borel subset A of X we have

$$\widetilde{\mu}_n(A) = \mathbb{P}(\xi_n \in A) = \mathbb{P}\big((\xi_n, \eta_n) \in A \times I\big) = \mu_n(A \times I),$$

where μ_n is a distribution of random vector (ξ_n, η_n). By Theorem 5.2 there exists a measure $\mu_* \in \mathcal{M}_1(X \times I)$ such that $\mu_n \to \mu_*$ weakly. Obviously $\widetilde{\mu}_n \to \widetilde{\mu}_*$ (weakly), where $\widetilde{\mu}_* \in \mathcal{M}_1(X)$ is given by

$$\widetilde{\mu}_*(A) = \mu_*(A \times I) \quad \text{for} \quad A \in \mathcal{B}(X). \quad \blacksquare$$

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REFERENCES

1. L. Arnold, *Random Dynamical Systems*, (Springer-Verlag, Berlin, 1998).
2. G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, (Cambridge University Press, Cambridge, 1992).
3. M. H. A. Davis, *Markov Models and Optimization*, (Chapman and Hall, London, 1993).
4. O. Diekmann, H. J. A. M. Heijmans and H. R. Thieme, On the stability of the cell size distribution, *J. Math. Biol.* **19**:227–248, (1984).
5. S. Ethier and T. Kurtz, *Markov Processes*, (John Wiley and Sons, New York, 1986).
6. R. Fortet and B. Mourier, Convergence de la répartition empirique vers la répartition théorétique, *Ann. Sci. École Norm. Sup*. **70**:267–285, 1953.
7. U. Frisch, Wave propagation in random media, in: *Probabilistic Methods in Applied Mathematics*, A. T. Bharucha-Reid, ed. (Academic Press, 1986).
8. B. Gaveau, T. Jacobson, M. Kac, and L. Schulman, Relativistic extension of the analogy between quantum mechanics and Brownian motion, *Phys. Rev. Lett*. **53**:419–422 (1984).
9. I. I. Gihman and A. Y. Skorohod, *Stochastic Differential Equations and their Applications* (Kiev, Naukova Dumka, 1982).
10. K. Horbacz, Randomly connected dynamical systems – asymptotic stability, *Ann. Pol. Math.* **68**1:31–50 (1998).
11. K. Horbacz, Random dynamical systems with jumps, *J. Appl. Probab*. **41**:890–910 (2004).
12. K. Horbacz, Randomly connected differential equations with Poisson type perturbations, *Nonlinear Stud.* **9.1**:81–98 (2002).

13. K. Horbacz and T. Szarek, Randomly connected dynamical systems on Banach space, *Stochas. Anal. Appl.* **19**:519–543 (2001).
14. J. B. Keller, Stochastic equations and wave propagation in random media, *Proc. Symp. Appl. Math.* **16**:1456–1470 (1964).
15. Y. Kifer, *Ergodic Theory of Random Transformations* (Birkhäuser, Basel, 1986).
16. J. F. C. Kingman, *Poisson Processes*, Oxford Studies in Probability, (Oxford University Press, New York, 1993).
17. T. Kudo and I. Ohba, Derivation of relativistic wave equation from the Poisson process, Pramana - *J. Phys.* **59**: 413–416 (2002).
18. A. Lasota, From fractals to stochastic differential equations, in *Chaos - The Interplay Between Stochastic and Deterministic Behaviour, Karpacz'95*, *Proceedings of the XXX-Ist Winter School of Theoretical Physics*, P. Garbaczewski, M. Wolf and A. Weron, eds. Lecture Notes in Physics, (Springer-Verlag, 1995) pp. 235–255
19. A. Lasota and M. C. Mackey, *Chaos, Fractals and Noise, Stochastic Aspects of Dynamics*, (Springer-Verlag, 1994).
20. A. Lasota and J. Myjak, Semifractals on Polish spaces, *Bull. Pol. Acad. Math.* **46**:179–196 (1998).
21. A. Lasota and T. Szarek, Dimension of measures invariant with respect to Ważewska partial differential equations, *J. Differential Equations* **196**(2):448–465 (2004).
22. A. Lasota and J. Traple, Invariant measures related with Poisson driven stochastic differential equation, *Stoch. Proc. Appl.* **106**(1)81–93 (2003).
23. A. Lasota and J. A. Yorke, Lower bound technique for Markov operators and iterated function systems, *Random Comput. Dynam.* **2**:41–77 (1994).
24. J. Malczak, Statistical stability of Poisson driven differential equation, *Bull. Pol. Acad. Math.* **41**:159–176 (1993).
25. S. Meyn and R. Tweedie, *Markov Chains and Stochastic Stability*, (Springer-Verlag, 1993).
26. M. I. Miller and D. L. Snyder, *Random Point Processes in Time and Space*, (Springer-Verlag, 1991).
27. J. Myjak and T. Szarek, Capacity of invariant measures related to Poisson-driven stochastic differential equations, *Nonlinearity* **16**:441–455 (2003).
28. G. Prodi, *Teoremi Ergodici per le Equazioni della Idrodinamica* (C.I.M.E., Roma, 1960).
29. T. Szarek, Invariant measures for Markov operators with applications to function systems, *Studia Math.* **154**(3):207–222 (2003).
30. T. Szarek and S. Wędrychowicz, Markov semigroups generated by a Poisson driven differential equation, *Nonlinear Anal. TMA* **50**(1):41–54 (2002).
31. J. Traple, Markov semigroups generated by Poisson driven differential equations, *Bull. Pol. Acad. Math.* **44**:161–182 (1996).